

ON THE C_k -STABLE CLOSURE OF THE CLASS OF (SEPARABLE) METRIZABLE SPACES

TARAS BANAKH AND SAAK GABRIELYAN

ABSTRACT. Denote by $\mathbf{C}_k[\mathfrak{M}]$ the C_k -stable closure of the class \mathfrak{M} of all metrizable spaces, i.e., $\mathbf{C}_k[\mathfrak{M}]$ is the smallest class of topological spaces that contains \mathfrak{M} and is closed under taking subspaces, homeomorphic images, countable topological sums, countable Tychonoff products, and function spaces $C_k(X, Y)$ with Lindelöf domain in this class. We show that the class $\mathbf{C}_k[\mathfrak{M}]$ coincides with the class of all topological spaces homeomorphic to subspaces of the function spaces $C_k(X, Y)$ with a separable metrizable space X and a metrizable space Y . We say that a topological space Z is Ascoli if every compact subset of $C_k(Z)$ is evenly continuous; by the Ascoli Theorem, each k -space is Ascoli. We prove that the class $\mathbf{C}_k[\mathfrak{M}]$ properly contains the class of all Ascoli \aleph_0 -spaces and is properly contained in the class of \mathfrak{P} -spaces, recently introduced by Gabrielyan and Kąkol. Consequently, an Ascoli space Z embeds into the function space $C_k(X, Y)$ for suitable separable metrizable spaces X and Y if and only if Z is an \aleph_0 -space.

1. INTRODUCTION

All topological spaces considered in this paper are regular T_1 -spaces. For two topological spaces X and Y , we denote by $C_k(X, Y)$ the space $C(X, Y)$ of all continuous functions from X into Y endowed with the compact-open topology. The space $C_k(X, \mathbb{R})$ of all real-valued functions on X is denoted by $C_k(X)$. In [19] Michael introduced the class of \aleph_0 -spaces, which appeared to be essential in studying function spaces $C_k(X, Y)$, see [17]. Recall that a topological space X is an \aleph_0 -space if X possesses a countable k -network. A family \mathcal{N} of subsets of X is a k -network in X if for any open subset $U \subset X$ and compact subset $K \subset U$ there exists a finite subfamily $\mathcal{F} \subset \mathcal{N}$ such that $K \subset \bigcup \mathcal{F} \subset U$. Any separable metrizable space is an \aleph_0 -space. In [19] Michael proved that for any \aleph_0 -spaces X and Y the function space $C_k(X, Y)$ is also an \aleph_0 -space. In particular, for any separable metrizable spaces X and Y the function space $C_k(X, Y)$ is not necessarily metrizable, but it is always an \aleph_0 -space.

Having in mind the Nagata-Smirnov metrization theorem, O'Meara [18] introduced the class of \aleph -spaces, which contains all metrizable spaces and all \aleph_0 -spaces. A topological space X is called an \aleph -space if X is regular and possesses a σ -locally finite k -network. It follows that a topological space is an \aleph_0 -space if and only if it is a Lindelöf \aleph -space. Foged [9] (and O'Meara [18]) generalized the result of Michael proving that for any Lindelöf \aleph -space X and any (paracompact) \aleph -space Y the function space $C_k(X, Y)$ is a (paracompact) \aleph -space. It is well-known (see [19, 15]) that the classes of \aleph -spaces and \aleph_0 -spaces are closed under taking subspaces, homeomorphic images, countable topological sums and countable Tychonoff products. These stability properties motivate the following definition.

Definition 1.1. A class \mathcal{X} of topological spaces is said to be

- stable if \mathcal{X} is closed under the operations of taking subspaces, homeomorphic images, countable topological sums, and countable Tychonoff products;
- C_k -stable if \mathcal{X} is stable and for any Lindelöf space $X \in \mathcal{X}$ and any space $Y \in \mathcal{X}$ the function space $C_k(X, Y)$ belongs to the class \mathcal{X} .

Let us observe that each stable class \mathcal{X} containing a non-empty topological space contains all zero-dimensional separable metrizable spaces, and if the closed interval $\mathbb{I} = [0, 1]$ belongs to \mathcal{X} , then \mathcal{X} contains all separable metrizable spaces (see Proposition 2.1 below).

For two classes \mathcal{X} and $\tilde{\mathcal{X}}$ of topological spaces, we shall say that $\tilde{\mathcal{X}}$ is an *extension* of \mathcal{X} if $\mathcal{X} \subset \tilde{\mathcal{X}}$. Among all (C_k) -stable extensions of \mathcal{X} there is the smallest one.

1991 *Mathematics Subject Classification.* 46E10 and 54C35 and 54E18.

The second named author was partially supported by Israel Science Foundation grant 1/12.

Definition 1.2. For a class \mathcal{X} of topological spaces, denote by $[\mathcal{X}]$ and $\mathbf{C}_k[\mathcal{X}]$ the smallest stable extension and the smallest C_k -stable extension of \mathcal{X} , respectively. The classes $[\mathcal{X}]$ and $\mathbf{C}_k[\mathcal{X}]$ will be called the stable closure and the C_k -stable closure of the class \mathcal{X} , respectively.

In the paper we study the C_k -stable closures $\mathbf{C}_k[\mathfrak{M}]$ and $\mathbf{C}_k[\mathfrak{M}_0]$ of the classes of all metrizable spaces \mathfrak{M} and all separable metrizable spaces \mathfrak{M}_0 , respectively.

For two classes \mathcal{X} and \mathcal{Y} of topological spaces we denote by $\mathbf{C}_k(\mathcal{X}, \mathcal{Y})$ the class of all topological spaces which embed into function spaces $C_k(X, Y)$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. In Section 2 we obtain the following characterization of the classes $\mathbf{C}_k[\mathfrak{M}]$ and $\mathbf{C}_k[\mathfrak{M}_0]$.

Theorem 1.3. $\mathbf{C}_k[\mathfrak{M}_0] = \mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M}_0)$ and $\mathbf{C}_k[\mathfrak{M}] = \mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M})$.

In other words, Theorem 1.3 means that the problem of whether a topological space Z belongs to the class $\mathbf{C}_k[\mathfrak{M}]$ (respectively, $\mathbf{C}_k[\mathfrak{M}_0]$) is equivalent to the embedding problem of the space Z into the function space $C_k(X, Y)$ with a separable metrizable space X and a (respectively, separable) metrizable space Y .

The aforementioned results of Michael imply that the class of \aleph_0 -spaces is a C_k -stable extension of the class \mathfrak{M}_0 . In [3] the first named author introduced another C_k -stable extension of the class \mathfrak{M}_0 , which consists of \mathfrak{P}_0 -spaces and is properly contained in the class of \aleph_0 -spaces. A regular topological space X is a \mathfrak{P}_0 -space if and only if X has a countable cp -network. A family \mathcal{N} of subsets of a topological space X is called a cp -network if for any point $x \in X$, each neighborhood $O_x \subset X$ of x and every subset $A \subset X$ containing x in its closure \bar{A} there exists a set $N \in \mathcal{N}$ such that $x \in N \subset O_x$ and moreover $N \cap A$ is infinite if A accumulates at x (see [3, 12]). So each space $X \in \mathbf{C}_k[\mathfrak{M}_0]$ is even a \mathfrak{P}_0 -space. However, there exists a \mathfrak{P}_0 -space X which does not belong to the class $\mathbf{C}_k[\mathfrak{M}]$ (see Example 6.4).

Following [12], we define a topological space Y to be a \mathfrak{P} -space if Y is regular and possesses a σ -locally finite cp -network. According to [12], this class is closed under taking subspaces, topological sums and countable Tychonoff products. Each \mathfrak{P} -space is an \aleph -space. A topological space X is a \mathfrak{P}_0 -space if and only if X is a Lindelöf \mathfrak{P} -space [12]. In Section 3 we prove the following theorem which gives a partial answer to Question 6.2 in [12].

Theorem 1.4. For any \aleph_0 -space X and each metrizable space Y the function space $C_k(X, Y)$ is a paracompact \mathfrak{P} -space.

The above discussion and Theorems 1.3 and 1.4 immediately imply

Corollary 1.5. The class $\mathbf{C}_k[\mathfrak{M}_0]$ is contained in the class of \mathfrak{P}_0 -spaces, and the class $\mathbf{C}_k[\mathfrak{M}]$ is contained in the class of paracompact \mathfrak{P} -spaces.

Since any \mathfrak{P} -space has countable tightness [12], we obtain that also all spaces in the class $\mathbf{C}_k[\mathfrak{M}]$ have countable tightness.

Another topological property which holds for every space in the class $\mathbf{C}_k[\mathfrak{M}]$ can be obtained by modifying the P -space property. Recall that a point x of a topological space X is called a P -point if for any countable family \mathcal{U} of neighborhoods of x the intersection $\bigcap \mathcal{U}$ is a neighborhood of x ; X is called a P -space if each $x \in X$ is a P -point. A point x of a topological space X is defined to be a P_{ω^1} -point if for any uncountable family \mathcal{U} of neighborhoods of x there is a countably infinite subfamily $\mathcal{V} \subset \mathcal{U}$ whose intersection $\bigcap \mathcal{V}$ is a neighborhood of x . A topological space X is called a P_{ω^1} -space if each its point is a P_{ω^1} -point. In Section 4 we prove the following theorem.

Theorem 1.6. For any separable metrizable space X and any metrizable space Y , the function space $C_k(X, Y)$ is a P_{ω^1} -space.

Now Theorems 1.3 and 1.6 imply

Corollary 1.7. Any space X in the class $\mathbf{C}_k[\mathfrak{M}]$ is a P_{ω^1} -space.

Corollaries 1.5 and 1.7 give us “upper” bounds for the class $\mathbf{C}_k[\mathfrak{M}]$. To obtain a “lower” bound for $\mathbf{C}_k[\mathfrak{M}]$, in Section 5 we introduce and study the class of Ascoli spaces. Recall that, for topological spaces X and Y , a subset \mathcal{K} of $C_k(X, Y)$ is *evenly continuous* if the map $(f, x) \mapsto f(x)$ is continuous as a map from $\mathcal{K} \times X$ to Y , i.e. for any $f \in \mathcal{K}$, $x \in X$ and neighborhood $O_{f(x)} \subset Y$ of $f(x)$ there exist neighborhoods $U_f \subset \mathcal{K}$ of f and $O_x \subset X$ of x such that $U_f(O_x) := \{g(y) : g \in U_f, y \in O_x\} \subset O_{f(x)}$. A topological space X is called an *Ascoli*

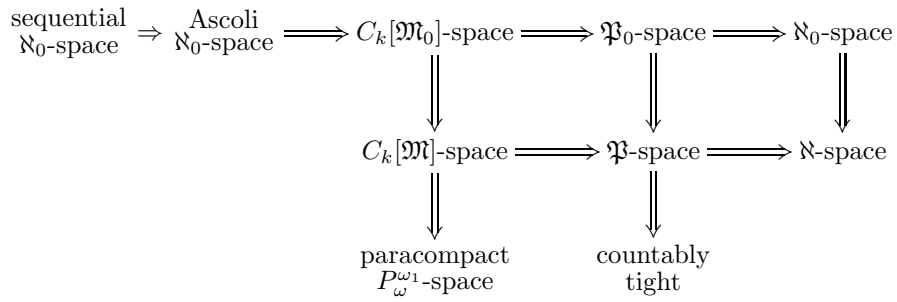
space if each compact subset \mathcal{K} of $C_k(X)$ is evenly continuous. By Ascoli's theorem [7, 3.4.20], each k -space, and hence each sequential space, is Ascoli.

Theorem 1.8. *Any Ascoli \aleph_0 -space belongs to the class $\mathbf{C}_k[\mathfrak{M}_0]$.*

If an Ascoli space Z embeds into some $C_k(X, Y)$ with $X, Y \in \mathfrak{M}_0$, then Z is an \aleph_0 -space by [19]. Conversely, if Z is an \aleph_0 -space, then $Z \in \mathbf{C}_k[\mathfrak{M}_0]$ by Theorem 1.8, and hence Z embeds into some $C_k(X, Y)$ with $X, Y \in \mathfrak{M}_0$ by Theorem 1.3. So we obtain

Corollary 1.9. *An Ascoli space Z embeds into a function space $C_k(X, Y)$ with $X, Y \in \mathfrak{M}_0$ if and only if Z is an \aleph_0 -space.*

Spaces that belong to the classes $\mathbf{C}_k[\mathfrak{M}_0]$ and $\mathbf{C}_k[\mathfrak{M}]$ will be called $C_k[\mathfrak{M}_0]$ -spaces and $C_k[\mathfrak{M}]$ -spaces, respectively. We summarize the obtained and known results in the following diagram.



Counterexamples constructed in the last section show that none of these implications can be reversed.

By analogy with the C_k -stable closure $\mathbf{C}_k[\mathcal{X}]$ of a class \mathcal{X} of topological spaces and being motivated also by the theory of Generalized Metric Spaces, in the next our paper [5] we introduce and characterize some natural types of the C_p -stable closures of the class \mathfrak{M}_0 .

2. CHARACTERIZATIONS OF THE CLASSES $\mathbf{C}_k[\mathfrak{M}_0]$ AND $\mathbf{C}_k[\mathfrak{M}]$

A topological space X *embeds* into a topological space Y if there exists a topological embedding $e : X \hookrightarrow Y$. Recall that a map $f : X \rightarrow Y$ between topological spaces is called *compact-covering* if for each compact subset $K \subset Y$ there is a compact subset $C \subset X$ such that $K = f(C)$.

Proposition 2.1. *If \mathcal{X} is a non-empty class of topological spaces, then*

- (1) $[\mathcal{X}]$ contains all zero-dimensional separable metrizable spaces;
- (2) $[\mathcal{X}]$ contains all separable metrizable spaces, provided that $\mathbb{I} \in \mathcal{X}$;
- (3) $C_k(Z, X) \in \mathbf{C}_k[\mathcal{X}]$ for every \aleph_0 -space Z and any $X \in \mathcal{X}$.

Proof. (1) The class \mathcal{X} is not empty and hence contains some topological space. Then its stable closure $[\mathcal{X}]$ contains the empty topological space and its 0-th power \emptyset^0 which is a singleton (this follows from the assumption that $[\mathcal{X}]$ is closed under taking countable Tychonoff products). Being closed under countable topological sums, the class $[\mathcal{X}]$ contains all countable discrete spaces, in particular, the doubleton $2 = \{0, 1\}$. By the countable productivity, $[\mathcal{X}]$ contains the Cantor cube 2^ω and all its subspaces. Since each zero-dimensional separable metrizable space embeds into the Cantor cube [16, 7.8], the class $[\mathcal{X}]$ contains all zero-dimensional separable metrizable spaces.

(2) If $\mathbb{I} \in \mathcal{X}$, then $\mathbb{I}^\mathbb{N} \in [\mathcal{X}]$ and hence $[\mathfrak{M}_0] \subseteq [\mathcal{X}]$ by [7, 4.2.10].

(3) The space Z , being an \aleph_0 -space, is the image of a separable metrizable space M under a compact-covering map $\xi : M \rightarrow Z$ (see [19]). Since every separable metrizable space is the image of a zero-dimensional separable metrizable space under a perfect (and hence compact-covering) map, without loss of generality we can suppose that the space M is zero-dimensional and hence belongs to the class $[\mathcal{X}]$ by the first statement. The C_k -stability of $\mathbf{C}_k[\mathcal{X}]$ guarantees that $C_k(M, X) \in \mathbf{C}_k[\mathcal{X}]$. The compact-covering property of the map ξ implies that the dual map $\xi^* : C_k(Z, X) \rightarrow C_k(M, X)$, $\xi^* : f \mapsto f \circ \xi$, is a topological embedding. Thus also $C_k(Z, X)$ belongs to $\mathbf{C}_k[\mathcal{X}]$. \square

Theorem 1.3 announced in the introduction is a partial case of the following two theorems.

Theorem 2.2. *The class $\mathbf{C}_k[\mathfrak{M}]$ coincides with the class $\mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M})$.*

Proof. The definition of the class $\mathbf{C}_k[\mathfrak{M}]$ guarantees that $\mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M}) \subset \mathbf{C}_k[\mathfrak{M}]$. By the minimality of $\mathbf{C}_k[\mathfrak{M}]$, the reverse inclusion will be proved as soon as we will check that the class $\mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M})$ is closed under taking countable topological sums, countable Tychonoff products and taking function spaces with Lindelöf domain.

To see that the class $\mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M})$ is closed under taking countable topological sums, we need to prove that for any non-empty spaces $X_n \in \mathfrak{M}_0$ and $Y_n \in \mathfrak{M}$, $n \in \omega$, the topological sum $\bigoplus_{n \in \omega} C_k(X_n, Y_n)$ embeds into the function space $C_k(X, Y)$ for some spaces $X \in \mathfrak{M}_0$ and $Y \in \mathfrak{M}$. Fix a singleton $*$. Now we consider the topological sums $X = \bigoplus_{n \in \omega} X_n$ and $Y = \{*\} \oplus \bigoplus_{n \in \omega} Y_n$ and the topological embedding

$$e : \bigoplus_{n \in \omega} C_k(X_n, Y_n) \hookrightarrow C_k(X, Y)$$

assigning to each function $f \in C_k(X_n, Y_n)$, $n \in \omega$, the function $\tilde{f} \in C_k(X, Y)$ defined by

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \in X_n; \\ *, & \text{if } x \in X_m \text{ and } m \neq n. \end{cases}$$

Since $X \in \mathfrak{M}_0$ and $Y \in \mathfrak{M}$, we conclude that $C_k(X, Y) \in \mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M})$, and hence $\bigoplus_{n \in \omega} C_k(X_n, Y_n) \in \mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M})$.

Next we prove that for any Lindelöf space $X \in \mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M})$ and any space $Y \in \mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M})$ the function space $C_k(X, Y)$ belongs to the class $\mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M})$. The Foged theorem [9] implies that the space X is an \aleph -space. Being Lindelöf, X is an \aleph_0 -space. By [19], the space X is the image of a separable metrizable space M under a compact-covering map. Hence, by Lemma 1 of [18], the function space $C_k(X, Y)$ embeds into the function space $C_k(M, Y)$. So, it is enough to prove that the function space $C_k(M, Y)$ belongs to the class $\mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M})$. By definition, the space Y embeds into the function space $C_k(A, B)$ for some spaces $A \in \mathfrak{M}_0$ and $B \in \mathfrak{M}$. Consequently, the function space $C_k(M, Y)$ embeds into the function space $C_k(M, C_k(A, B))$. By Theorem 3.4.9 of [7], the map $C_k(M, C_k(A, B)) \rightarrow C_k(M \times A, B)$ assigning to each function $f : M \rightarrow C_k(A, B)$ the function $\tilde{f} : M \times A \rightarrow B$, $\tilde{f} : (m, a) \mapsto f(m)(a)$, is a homeomorphism. Since $M \times A$ is a separable metrizable space, the function space $C_k(M \times A, B)$ belongs to the class $\mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M})$, and hence the spaces $C_k(M, C_k(A, B))$, $C_k(M, Y)$ and $C_k(X, Y)$ also belong to $\mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M})$.

Finally we show that the class $\mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M})$ is closed under taking countable Tychonoff products. Fix any spaces $X_n \in \mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M})$, $n \in \omega$. As the class $\mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M})$ is closed under taking countable topological sums, the topological sum $X = \bigoplus_{n \in \omega} X_n$ belongs to the class $\mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M})$. Since the class $\mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M})$ is closed also under taking function spaces with Lindelöf domain, the function space $C_k(\omega, X)$ belongs to the class $\mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M})$. Taking into account that $\prod_{n \in \omega} X_n \subset X^\omega = C_k(\omega, X) \in \mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M})$, we conclude that $\prod_{n \in \omega} X_n \in \mathbf{C}_k(\mathfrak{M}_0, \mathfrak{M})$. \square

Below we characterize the class $\mathbf{C}_k[\mathfrak{M}_0]$.

Theorem 2.3. *For a topological space X the following conditions are equivalent:*

- (1) X belongs to the class $\mathbf{C}_k[\mathfrak{M}_0]$;
- (2) X is Lindelöf and belongs to the class $\mathbf{C}_k[\mathfrak{M}]$;
- (3) X embeds into the function space $C_k(Z, Y)$ for some spaces $Z, Y \in \mathfrak{M}_0$;
- (4) X embeds into the function space $C_k(Z)$ for some zero-dimensional space $Z \in \mathfrak{M}_0$.

Proof. (1) \Rightarrow (2): As $\mathbf{C}_k[\mathfrak{M}_0] \subset \mathbf{C}_k[\mathfrak{M}]$, we have also $X \in \mathbf{C}_k[\mathfrak{M}]$. The space X , being an \aleph_0 -space, is Lindelöf by [19].

(2) \Rightarrow (3): Assume that X is a Lindelöf space from the class $\mathbf{C}_k[\mathfrak{M}]$. By Theorem 2.2, X embeds into the function space $C_k(Z, Y)$ for some spaces $Z \in \mathfrak{M}_0$ and $Y \in \mathfrak{M}$. The Foged theorem [9] implies that the space X is an \aleph -space. Being Lindelöf, X is an \aleph_0 -space (see [13]). By [19], X is the image of a separable metrizable space A under a compact-covering continuous map $\xi : A \rightarrow X \subset C_k(Z, Y)$. Consider the map $\varphi : A \times Z \rightarrow Y$ assigning to each pair $(a, z) \in A \times Z$ the value $\xi(a)(z)$ of the function $\xi(a)$ at the point z . We claim that this map is continuous at each point $(a, z) \in A \times Z$. Take any sequence $((a_k, z_k))_{k \in \omega}$ in $A \times Z$ converging to (a, z) in the separable metrizable space $A \times Z$, and any neighborhood $O_{\varphi(a, z)} \subset Y$ of the point $\varphi(a, z)$. For every

$k \in \omega$ consider the function $f_k := \xi(a_k) \in X \subset C_k(Z, Y)$ and observe that the sequence $\{f_k\}_{k \in \omega}$ converges to the function $f := \xi(a)$. By Ascoli's theorem [7, 3.4.20], the compact subset $\mathcal{K} := \{f\} \cup \{f_k\}_{k \in \omega} \subset C_k(Z, Y)$ is evenly continuous, that allows us to find a neighborhood U_f of f in \mathcal{K} and a neighborhood O_z of z such that $U_f(O_z) \subset O_{\varphi(a, z)}$. By the continuity of the map ξ , there is a number $k_0 \in \omega$ such that $\xi(a_k) \in U_f$ for all $k \geq k_0$. Then for any $k \geq k_0$ with $z_k \in O_z$ we get $\varphi(a_k, z_k) = f_k(z_k) \in U_f(O_z) \subset O_{\varphi(a, z)}$. So, the map $\varphi : A \times Z \rightarrow Y$ is continuous, and hence its image $Y_0 := \varphi(A \times Z) \subset Y$ is separable. Then $X \subset C_k(Z, Y_0) \subset C_k(Z, Y)$, where the spaces Z and Y_0 are separable and metrizable.

(3) \Rightarrow (4): Assume that X embeds into the function space $C_k(A, B)$ for some separable metrizable spaces A and B . Since the separable metrizable space B embeds into the countable product \mathbb{R}^ω , the function space $C_k(A, B)$ embeds into the function space $C_k(A, \mathbb{R}^\omega)$, which is homeomorphic to the function space $C_k(A \times \omega, \mathbb{R}) = C_k(A \times \omega)$. The separable metrizable space $A \times \omega$ can be written as the image of a zero-dimensional separable metrizable space Z under a compact-covering map $\eta : Z \rightarrow A \times \omega$. Then the dual map $\eta^* : C_k(A \times \omega) \rightarrow C_k(Z)$, $\eta^* : f \mapsto f \circ \eta$, is a topological embedding (see Lemma 1 in [18]). Thus

$$X \hookrightarrow C_k(A, B) \hookrightarrow C_k(A, \mathbb{R}^\omega) = C_k(A \times \omega) \hookrightarrow C_k(Z).$$

(4) \Rightarrow (1): Assume that X embeds into the function space $C_k(Z)$ for some zero-dimensional separable metrizable space Z . Since $Z, \mathbb{R} \in \mathfrak{M}_0$, the function space $C_k(Z) = C_k(Z, \mathbb{R})$ as well as its subspace X belong to the class $\mathbf{C}_k[\mathfrak{M}_0]$. \square

3. PROOF OF THEOREM 1.4

In this section we prove a more general result than Theorem 1.4. For this purpose we need some definitions.

A family \mathcal{I} of compact subsets of a topological space X is called an *ideal of compact sets* if $\bigcup \mathcal{I} = X$ and for any sets $A, B \in \mathcal{I}$ and any compact subset $K \subset X$ we get $A \cup B \in \mathcal{I}$ and $A \cap K \in \mathcal{I}$, i.e. if \mathcal{I} covers X and is closed under taking finite unions and closed subspaces.

For any two topological spaces X and Y , each ideal \mathcal{I} of compact subsets of X determines the \mathcal{I} -open topology on the space $C(X, Y)$ of continuous functions from X to Y . A subbase of this topology consists of the sets

$$[K; U] = \{f \in C(X, Y) : f(K) \subset U\},$$

where $K \in \mathcal{I}$ and U is an open subset of Y . The space $C(X, Y)$ endowed with the \mathcal{I} -open topology will be denoted by $C_{\mathcal{I}}(X, Y)$. So, $C_k(X, Y) = C_{\mathcal{I}}(X, Y)$ for the ideal \mathcal{I} of all compact subsets of X . For the ideal \mathcal{I} of all finite subsets of X , the function space $C_{\mathcal{I}}(X, Y)$ is denoted by $C_p(X, Y)$.

We are interested in detecting \mathfrak{P} -spaces among the function spaces $C_{\mathcal{I}}(X, Y)$.

Definition 3.1. An ideal \mathcal{I} of compact subsets of a topological space X is called *discretely-complete* if for any compact subsets $A, B \subset X$ with countable discrete difference $B \setminus A$, the inclusion $A \in \mathcal{I}$ implies $B \in \mathcal{I}$.

Since the ideal of all compact subsets of a topological space is trivially discretely-complete, the following theorem implies Theorem 1.4.

Theorem 3.2. For any discretely-complete ideal \mathcal{I} of compact subsets of an \aleph_0 -space X and any metrizable space Y the function space $C_{\mathcal{I}}(X, Y)$ is a paracompact \mathfrak{P} -space. If Y is separable, then $C_{\mathcal{I}}(X, Y)$ is a \mathfrak{P}_0 -space.

Proof. For the \aleph_0 -space X fix a countable k -network \mathcal{K} , which is closed under taking finite unions and finite intersections. For the metrizable space Y fix a σ -locally finite base of the topology $\mathcal{D} = \bigcup_{j \in \omega} \mathcal{D}_j$ (which exists by the Nagata-Smirnov metrization theorem [7, 4.4.7]). We claim that the family

$$[\mathcal{K}; \mathcal{D}] = \{[K_1; D_1] \cap \cdots \cap [K_n; D_n] : K_1, \dots, K_n \in \mathcal{K}, D_1, \dots, D_n \in \mathcal{D}\}$$

is a σ -locally finite cp -network in $C_{\mathcal{I}}(X, Y)$.

To see that the family $[\mathcal{K}; \mathcal{D}]$ is σ -locally finite in $C_{\mathcal{I}}(X, Y)$, write it as

$$[\mathcal{K}; \mathcal{D}] = \bigcup_{n \in \mathbb{N}} \bigcup_{K_1, \dots, K_n \in \mathcal{K}} \bigcup_{(l_1, \dots, l_n) \in \omega^n} [[K_1; \mathcal{D}_{l_1}] \wedge \cdots \wedge [K_n; \mathcal{D}_{l_n}]],$$

where

$$[[K_1; \mathcal{D}_{l_1}] \wedge \cdots \wedge [K_n; \mathcal{D}_{l_n}]] := \{[K_1; D_1] \cap \cdots \cap [K_n; D_n] : D_1 \in \mathcal{D}_{l_1}, \dots, D_n \in \mathcal{D}_{l_n}\}.$$

We claim that for any $K \in \mathcal{K}$ and $l \in \omega$ the family $\llbracket K; \mathcal{D}_l \rrbracket$ is locally finite in $C_{\mathcal{I}}(X, Y)$. Given any function $f \in C_{\mathcal{I}}(X, Y)$, choose a point $x \in K$. As $\bigcup \mathcal{I} = X$, the singleton $\{x\}$ belongs to the ideal \mathcal{I} . Since the family \mathcal{D}_l is locally finite in Y , the point $f(x) \in Y$ has a neighborhood $V \subset Y$ meeting only finitely many members of \mathcal{D}_l . Then the open neighborhood $\llbracket \{x\}; V \rrbracket \subset C_{\mathcal{I}}(X, Y)$ of f meets only those elements $\llbracket K; D \rrbracket \in \llbracket K; \mathcal{D}_l \rrbracket$ for which D intersects V . By the choice of V , the number of such elements is finite. So, the family $\llbracket K; \mathcal{D}_l \rrbracket$ is locally finite and therefore so is the family $\llbracket K_1; \mathcal{D}_{l_1} \rrbracket \wedge \cdots \wedge \llbracket K_n; \mathcal{D}_{l_n} \rrbracket$ for every sets $K_1, \dots, K_n \in \mathcal{K}$ and every numbers $l_1, \dots, l_n \in \omega$.

Now we prove that the family $\llbracket \mathcal{K}; \mathcal{D} \rrbracket$ is a *cp*-network for $C_{\mathcal{I}}(X, Y)$. Fix any function $f \in C_{\mathcal{I}}(X, Y)$, an open neighborhood $O_f \subset C_{\mathcal{I}}(X, Y)$ of f and a subset $A \subset C_{\mathcal{I}}(X, Y)$ containing f in its closure. We lose no generality assuming that the neighborhood O_f is of basic form

$$O_f = [C_1; U_1] \cap \cdots \cap [C_n; U_n]$$

for some compact sets $C_1, \dots, C_n \in \mathcal{I}$ and some open sets $U_1, \dots, U_n \in \mathcal{D}$.

For every $i \in \{1, \dots, n\}$, consider the countable family

$$\mathcal{K}_i := \{K \in \mathcal{K} : C_i \subset K \subset f^{-1}(U_i)\},$$

and let $\mathcal{K}_i = \{K'_{i,j}\}_{j \in \omega}$ be its enumeration. For every $j \in \mathbb{N}$ we put $K_{i,j} := \bigcap_{k \leq j} K'_{i,k}$. It follows that the decreasing sequence $\{K_{i,j}\}_{j \in \omega}$ converges to C_i in the sense that each open neighborhood of C_i contains all but finitely many sets $K_{i,j}$. Then the sets

$$\mathcal{F}_j := \bigcap_{i=1}^n [K_{i,j}; U_i] \in \llbracket \mathcal{K}; \mathcal{D} \rrbracket, \quad j \in \omega,$$

form an increasing sequence of sets in the function space $C_k(X, Y)$. We claim that

$$\bigcap_{i=1}^n [C_i; U_i] = O_f = \bigcup_{j \in \omega} \mathcal{F}_j.$$

Suppose for a contradiction that there exists a function $g \in \bigcap_{i=1}^n [C_i; U_i]$ which does not belong to $\bigcup_{j \in \mathbb{N}} \mathcal{F}_j$. Then for every $j \in \omega$ we can find an index $i_j \in \{1, \dots, n\}$ such that $g \notin [K_{i_j,j}; U_{i_j}]$. This means that $g(x_j) \notin U_{i_j}$ for some point $x_j \in K_{i_j,j}$. By the Pigeonhole Principle, there is $m \in \{1, \dots, n\}$ such that the set $J_m := \{j \in \mathbb{N} : i_j = m\}$ is infinite. As the decreasing sequence $\{K_{m,j}\}_{j \in J_m}$ converges to the compact set C_m , the set $C_m \cup \{x_j\}_{j \in J_m}$ is compact.

Since any compact subset of the \aleph_0 -space X is metrizable, we can find an infinite subset J' of J_m such that the sequence $\{x_j\}_{j \in J'}$ converges to some point $x_0 \in C_m$. As g is continuous, the sequence $\{g(x_j)\}_{j \in J'}$ converges to the point $g(x_0) \in g(C_m) \subset U_m$, and hence we can assume also that $g(x_j) \in U_m$ for every $j \in J'$. But this contradicts the choice of the points x_j and therefore proves the equality $O_f = \bigcup_{j \in \omega} \mathcal{F}_j$.

It follows that $f \in \mathcal{F}_j \subset O_f$ for some $j \in \omega$, which means that the family $\llbracket \mathcal{K}; \mathcal{D} \rrbracket$ is a network in the function space $C_{\mathcal{I}}(X, Y)$. So without loss of generality we can assume that $f \in \mathcal{F}_j$ for every $j \in \omega$.

Now, assuming that the set A accumulates at f , we shall prove that for some $j \in \omega$ the intersection $\mathcal{F}_j \cap A$ is infinite. Replacing A by $O_f \cap A$, we can assume that $A \subset O_f$. Suppose for a contradiction that for every $j \in \omega$ the intersection $A_j := \mathcal{F}_j \cap A$ is finite. Then $A = A \cap O_f = \bigcup_{j \in \omega} A_j$ is the countable union of the increasing sequence $\{A_j\}_{j \in \omega}$ of finite subsets of $C_{\mathcal{I}}(X, Y)$.

For every function $\alpha \in A \setminus A_0$ we denote by j_α the unique natural number such that $\alpha \in A_{j_\alpha+1} \setminus A_{j_\alpha} = A_{j_\alpha+1} \setminus \mathcal{F}_{j_\alpha}$. Since $\alpha \notin \mathcal{F}_{j_\alpha} = \bigcap_{i=1}^n [K_{i,j_\alpha}; U_i]$, there is an index $i_\alpha \in \{1, \dots, n\}$ such that $\alpha \notin [K_{i_\alpha,j_\alpha}; U_{i_\alpha}]$ and a point $x_\alpha \in K_{i_\alpha,j_\alpha}$ such that $\alpha(x_\alpha) \notin U_{i_\alpha}$.

For every $i \in \{1, \dots, n\}$ consider the subsequence

$$A(i) := \{\alpha \in A \setminus A_0 : i_\alpha = i\}$$

and observe that $A \setminus A_0 = \bigcup_{i=1}^n A(i)$. So, there exists $s \in \{1, \dots, n\}$ such that the set $A(s)$ is infinite and $f \in \overline{A(s)}$.

Taking into account that the sequence of sets $\{K_{s,j}\}_{j \in \omega}$ converges to the compact set C_s , we can find a finite subset B of $A(s)$ such that $f(x_\alpha) \in U_s$ for every $\alpha \in A(s) \setminus B$. Put

$$A' := A(s) \setminus B \quad \text{and} \quad C' := C_s \cup \{x_\alpha\}_{\alpha \in A'}$$

and observe that the compact set C' belongs to the ideal \mathcal{I} by the discrete-completeness of \mathcal{I} . Since $[C'; U_s]$ is an open neighborhood of $f \in \overline{A'}$, there is a function $\alpha \in A' \cap [C'; U_s]$, which is not possible as $\alpha(x_\alpha) \notin U_s$. Thus $\mathcal{F}_j \cap A$ is infinite for some $j \in \omega$. Therefore $[\mathcal{K}; \mathcal{D}]$ is a cp -network in $C_{\mathcal{I}}(X, Y)$ and hence $C_{\mathcal{I}}(X, Y)$ is a \mathfrak{P} -space.

The paracompactness of the space $C_{\mathcal{I}}(X, Y)$ was implicitly proved in Lemmas 5 and 6 of [18].

If the space Y is separable, then the family \mathcal{D} is countable and so is the cp -network $[\mathcal{K}; \mathcal{D}]$. This means that the function space $C_{\mathcal{I}}(X, Y)$ is a \mathfrak{P}_0 -space. \square

4. DETECTING FUNCTION SPACES WHICH ARE $P_{\omega}^{\omega_1}$ -SPACES

In this section we discuss a topological property that allows us to construct an example of a \mathfrak{P}_0 -space, which does not belong to the class $\mathbf{C}_k[\mathfrak{M}]$ (see Example 6.4). This property is a modification of the P -space property, and its study was initiated by Greinecker and Ravsky in <http://math.stackexchange.com/questions/377038>.

Definition 4.1. A point x of a topological space X is called a P_{λ}^{κ} -point for cardinals $\lambda \leq \kappa$ if for any family $(U_{\alpha})_{\alpha \in \kappa}$ of neighborhoods of x there is a subset $\Lambda \subset \kappa$ of cardinality $|\Lambda| = \lambda$ such that $\bigcap_{\alpha \in \Lambda} U_{\alpha}$ is a neighborhood of x . The space X is called a P_{λ}^{κ} -space if each $x \in X$ is a P_{λ}^{κ} -point.

The next proposition implies that a topological space X is a P -space if and only if it is a P_{ω}^{ω} -space.

Proposition 4.2. A point x of a topological space X is a P -point if and only if it is a P_{ω}^{ω} -point.

Proof. The “only if” part of this statement is trivial. To prove the “if” part, assume that x is a P_{ω}^{ω} -point. To prove that x is a P -point, take any sequence $(U_n)_{n \in \omega}$ of neighborhoods of x . For every $n \in \omega$, put $V_n = \bigcap_{i \leq n} U_i$. Since x is a P_{ω}^{ω} -point, for the sequence of neighborhoods $(V_n)_{n \in \omega}$ of x there is an infinite subset $\Lambda \subset \omega$ such that $\bigcap_{n \in \Lambda} V_n$ is a neighborhood of x . Since the sequence $(V_n)_{n \in \omega}$ is decreasing, we conclude that $\bigcap_{n \in \omega} U_n = \bigcap_{n \in \omega} V_n = \bigcap_{n \in \Lambda} V_n$ is a neighborhood of x . So, x is a P -point in X . \square

For a point x of a topological space X , denote by $\chi(x, X)$ the *character* of X at x , i.e., the smallest cardinality of a neighborhood base at x .

Proposition 4.3. A point x of a topological space X is a P_{λ}^{κ} -point for any cardinals $\lambda < \kappa$ with $\kappa > \chi(x, X)$.

Proof. Fix a neighborhood base \mathcal{B}_x at x of cardinality $|\mathcal{B}_x| = \chi(x, X)$. To show that x is a P_{λ}^{κ} -point for cardinals $\lambda < \kappa > \chi(x, X)$, take any family of neighborhoods $(U_{\alpha})_{\alpha \in \kappa}$ of x . For each $\alpha \in \kappa$ find a basic set $B_{\alpha} \in \mathcal{B}_x$ which is contained in the neighborhood U_{α} . Since $|\mathcal{B}_x| = \chi(x, X) < \kappa$, by the Pigeonhole Principle, there is $B \in \mathcal{B}_x$ such that the set $\Lambda = \{\alpha \in \kappa : B_{\alpha} = B\}$ has cardinality $|\Lambda| > \lambda$. Then $\bigcap_{\alpha \in \Lambda} U_{\alpha} \supset B$ is a neighborhood of x . \square

By Proposition 4.3, each first countable space is a P_{ω}^{κ} -space for any uncountable cardinal κ .

To detect function spaces $C_{\mathcal{I}}(X, Y)$ which are P_{λ}^{κ} -spaces, let us introduce the corresponding property for ideals of compact sets.

Definition 4.4. An ideal \mathcal{I} of compact subsets of a topological space X is called a P_{λ}^{κ} -ideal for cardinals $\lambda \leq \kappa$ if for any family of compact subsets $\{K_{\alpha}\}_{\alpha \in \kappa} \subset \mathcal{I}$ there exists a subset $\Lambda \subset \kappa$ of cardinality $|\Lambda| = \lambda$ such that the union $\bigcup_{\alpha \in \Lambda} K_{\alpha}$ is contained in some compact set $K \in \mathcal{I}$.

Theorem 4.5. Let $\lambda \leq \kappa$ be two cardinals with $\text{cf}(\kappa) > \omega$. For an ideal \mathcal{I} of compact subsets of a Tychonoff space X the following conditions are equivalent:

- (1) \mathcal{I} is a P_{λ}^{κ} -ideal;
- (2) for every metrizable space Y the function space $C_{\mathcal{I}}(X, Y)$ is a P_{λ}^{κ} -space;
- (3) the function space $C_{\mathcal{I}}(X)$ is a P_{λ}^{κ} -space.

Proof. (1) \Rightarrow (2): Assume that \mathcal{I} is a P_{λ}^{κ} -ideal. We need to show that for any metrizable space Y the function space $C_{\mathcal{I}}(X, Y)$ is a P_{λ}^{κ} -space. Since each metrizable space embeds into a Banach space, we can assume that Y is a Banach space endowed with a norm $\|\cdot\|$. Then $C_{\mathcal{I}}(X, Y)$ is a linear topological space. So it suffices to check that the constant zero function $\mathbf{0} : X \rightarrow \{0\} \subset Y$ is a P_{λ}^{κ} -point of the function space $C_{\mathcal{I}}(X, Y)$. Fix a family $(U_{\alpha})_{\alpha \in \kappa}$ of neighborhoods of $\mathbf{0}$ in $C_{\mathcal{I}}(X, Y)$. By the definition of the topology of the function space $C_{\mathcal{I}}(X, Y)$,

for every $\alpha \in \kappa$ there is a compact set $K_\alpha \in \mathcal{I}$ and a positive real number ε_α such that the neighborhood U_α contains the basic neighborhood

$$[K_\alpha; \varepsilon_\alpha] := \{f \in C_{\mathcal{I}}(X, Y) : \sup_{x \in K_\alpha} \|f(x)\| < \varepsilon_\alpha\}.$$

Since the cardinal κ has uncountable cofinality, there is a positive real number ε such that the set $\Omega = \{\alpha \in \kappa : \varepsilon_\alpha \geq \varepsilon\}$ has cardinality $|\Omega| = \kappa$. Since \mathcal{I} is a P_λ^κ -ideal, for the family of compact sets $\{K_\alpha\}_{\alpha \in \Omega} \subset \mathcal{I}$ there is a subset $\Lambda \subset \Omega$ of cardinality $|\Lambda| = \lambda$ such that the union $\bigcup_{\alpha \in \Lambda} K_\alpha$ is contained in some compact set $K \in \mathcal{I}$. Then the intersection $\bigcap_{\alpha \in \Lambda} U_\alpha$ contains the neighborhood $[K; \varepsilon]$ of $\mathbf{0}$, which means that $\mathbf{0}$ is a P_λ^κ -point of $C_{\mathcal{I}}(X, Y)$.

The implication (2) \Rightarrow (3) is trivial. To prove that (3) \Rightarrow (1), assume that the function space $C_{\mathcal{I}}(X)$ is a P_λ^κ -space. We have to prove that \mathcal{I} is a P_λ^κ -ideal. Fix any family of compact sets $\{K_\alpha\}_{\alpha \in \kappa} \subset \mathcal{I}$ and for every $\alpha \in \kappa$ consider the neighborhood

$$U_\alpha = \{f \in C_{\mathcal{I}}(X) : \sup_{x \in K_\alpha} |f(x)| < 1\}$$

of the constant zero function $\mathbf{0} \in C_{\mathcal{I}}(X)$. Since $C_{\mathcal{I}}(X)$ is a P_λ^κ -space, there exists a subset $\Lambda \subset \kappa$ of cardinality $|\Lambda| = \lambda$ such that the intersection $\bigcap_{\alpha \in \Lambda} U_\alpha$ is a neighborhood of $\mathbf{0}$. Then this intersection contains a basic neighborhood $[K; \varepsilon]$ for some compact set $K \in \mathcal{I}$ and some $\varepsilon > 0$. We claim that $\bigcup_{\alpha \in \Lambda} K_\alpha \subset K$. Indeed, assuming the converse we can find $\alpha \in \Lambda$ and a point $x \in K_\alpha \setminus K$. Since the space X is Tychonoff, there exists a continuous function $f : X \rightarrow \mathbb{R}$ such that $f(K) \subset \{0\}$ and $f(x) = 1$. Then $f \in [K; \varepsilon] \not\subset U_\alpha$ that contradicts the choice of $[K; \varepsilon]$. \square

Since the ideal of finite subsets of an uncountable topological space fails to be a $P_\omega^{\omega_1}$ -ideal, Theorem 4.5 implies:

Corollary 4.6. *For any uncountable space Tychonoff space X the function space $C_p(X)$ fails to be a $P_\omega^{\omega_1}$ -space.*

An ideal \mathcal{I} of compact subsets of a topological space X is called a σ -ideal if each compact subset $K \subset X$, which can be written as the countable union $K = \bigcup_{n \in \omega} K_n$ of compact sets $K_n \in \mathcal{I}$, $n \in \omega$, belong to the ideal \mathcal{I} . For example, for any infinite cardinal κ the ideal of all compact subsets of cardinality $\leq \kappa$ in a topological space X is a σ -ideal.

Now we see that Theorem 1.6 is a partial case of the following theorem.

Theorem 4.7. *Any σ -ideal \mathcal{I} of compact subsets of a separable metrizable space X is a $P_\omega^{\omega_1}$ -ideal. Consequently, for every metrizable space Y the function space $C_{\mathcal{I}}(X, Y)$ is a $P_\omega^{\omega_1}$ -space.*

Proof. To show that \mathcal{I} is a $P_\omega^{\omega_1}$ -ideal, fix any indexed family $\mathcal{K} = \{K_\alpha\}_{\alpha < \omega_1}$ of non-empty compact sets in \mathcal{I} . If the set \mathcal{K} is countable, then for some compact set $K \in \mathcal{K}$ the set $\Lambda = \{\alpha \in \kappa : K_\alpha = K\}$ is infinite and the union $\bigcup_{\alpha \in \Lambda} K_\alpha = K$ belongs to \mathcal{I} . So, we assume that \mathcal{K} is uncountable. The set \mathcal{K} can be considered as an uncountable subset of the hyperspace $\exp(X)$ of all non-empty compact sets endowed with the Vietoris topology. It is well-known that the hyperspace $\exp(X)$ of the separable metrizable space X is separable and metrizable (by the Hausdorff metric). Since separable metrizable spaces do not contain uncountable discrete subspaces, the uncountable set $\mathcal{K} \subset \exp(X)$ is not discrete and hence contains a non-trivial convergent sequence with the limit point. So we can find a sequence of pairwise distinct countable ordinals $(\alpha_n)_{n \in \omega}$ such that the sequence $(K_{\alpha_n})_{n \in \omega}$ converges to some compact set $K_\infty \in \mathcal{K}$ in the hyperspace $\exp(X)$. It follows that the union

$$K = K_\infty \cup \bigcup_{n \in \omega} K_{\alpha_n}$$

is a compact set in X . Since \mathcal{I} is a σ -ideal, the compact set K belongs to the ideal \mathcal{I} and contains the union $\bigcup_{n \in \omega} K_{\alpha_n}$ witnessing that \mathcal{I} is a $P_\omega^{\omega_1}$ -ideal.

By Theorem 4.5, for every metrizable space Y the function space $C_{\mathcal{I}}(X, Y)$ is a $P_\omega^{\omega_1}$ -space. \square

5. ASCOLI SPACES

The results of this section are motivated by the problem of detecting spaces that necessarily belong to the C_k -stable closure $\mathbf{C}_k[\mathcal{X}]$ of any non-empty class of topological spaces \mathcal{X} . To detect spaces that embed into some nice function spaces we shall exploit the construction of the canonical map.

For any topological spaces X and Y the canonical map

$$\delta : X \rightarrow C_k(C_k(X, Y), Y)$$

assigns to each point $x \in X$ the Y -valued Dirac measure $\delta_x : C_k(X, Y) \rightarrow Y$ concentrated at x . The Y -valued Dirac measure δ_x assigns to each function $f \in C_k(X, Y)$ its value $f(x)$ at x ; so $\delta_x(f) = f(x)$. The function $\delta_x : C_k(X, Y) \rightarrow Y$ is continuous at each function $f \in C_k(X, Y)$ since for any open neighborhood $V \subset Y$ of $\delta_x(f) = f(x)$, the set $[\{x\}; V]$ is an open neighborhood of f in $C_k(X, Y)$ with $\delta_x([\{x\}; V]) \subset V$. This shows that the canonical map $\delta : X \rightarrow C_k(C_k(X, Y), Y)$ is well-defined.

We are interested in detecting topological spaces X and Y for which the canonical map $\delta : X \rightarrow C_k(C_k(X, Y), Y)$ is a topological embedding. We start with finding conditions implying the continuity of the canonical map δ .

Recall that a topological space X is called

- *sequential* if for each non-closed subset $A \subset X$ there is a sequence $\{a_n\}_{n \in \omega} \subset A$ converging to some point $a \in \bar{A} \setminus A$;
- a *k-space* if for each non-closed subset $A \subset X$ there is a compact subset $K \subset X$ such that $A \cap K$ is not closed in K ;
- a *$k_{\mathbb{R}}$ -space* if a real-valued function f on X is continuous if and only if its restriction $f|_K$ to any compact subset K of X is continuous.

In the next definition we generalize the notions of a $k_{\mathbb{R}}$ -space and of the Ascoli space defined in Introduction.

Definition 5.1. *Let Y be a topological space. A topological space X is called*

- a *k_Y -space* if for any discontinuous function $f : X \rightarrow Y$ there is a compact subset $K \subset X$ such that the restriction $f|_K$ is discontinuous;
- a *Y -Ascoli space* if each compact subset $\mathcal{K} \subset C_k(X, Y)$ is evenly continuous.

So, X is an Ascoli space if and only if X is \mathbb{R} -Ascoli. By Ascoli's theorem [7, 3.4.20], each k -space is Y -Ascoli for any regular space Y . On the other hand, Noble [21] proved that each $k_{\mathbb{R}}$ -space is Y -Ascoli for any Tychonoff space Y . By Corollary 5.3 proved below, each Ascoli space is 2-Ascoli, where $2 = \{0, 1\}$ is the doubleton endowed with the discrete topology. Therefore we have the following implications:

$$\text{sequential} \Rightarrow k\text{-space} \Rightarrow k_{\mathbb{R}}\text{-space} \Rightarrow \text{Ascoli} \Rightarrow 2\text{-Ascoli}.$$

Now we prove some elementary properties of Y -Ascoli spaces. Let us recall that a subspace Z of a topological space X is a *retract* of X if there is a continuous map $r : X \rightarrow Z$ such that $r(z) = z$ for all $z \in Z$.

Proposition 5.2. *Let Y be a topological space.*

- (1) *If X is a Y -Ascoli space, then each retract Z of X is Y -Ascoli.*
- (2) *For any family $\{X_i\}_{i \in I}$ of Y -Ascoli spaces the topological sum $X := \bigoplus_{i \in I} X_i$ is Y -Ascoli.*
- (3) *Each Y -Ascoli space X is Y^{κ} -Ascoli for every cardinal κ .*
- (4) *Each Y -Ascoli space X is Z -Ascoli for every subspace $Z \subset Y$.*

Proof. (1) Let Z be a retract of a Y -Ascoli space X and $r : X \rightarrow Z$ be a retraction of X onto Z . This retraction induces a continuous extension operation $r^* : C_k(Z, Y) \rightarrow C_k(X, Y)$, $r^* : f \mapsto f \circ r$. To show that the space Z is Y -Ascoli, fix a compact subspace $\mathcal{K} \subset C_k(Z, Y)$, a function $f \in \mathcal{K}$, a point $z \in Z$ and a neighborhood $O_{f(z)} \subset Y$ of $f(z)$. Consider the function $f \circ r : X \rightarrow Y$ and the compact subset $r^*(\mathcal{K}) \subset C_k(X, Y)$. Since the space X is Y -Ascoli, there are a neighborhood $O_z \subset X$ of z and a neighborhood $U_{f \circ r} \subset r^*(\mathcal{K}) \subset C_k(X, Y)$ of $f \circ r$ such that $U_{f \circ r}(O_z) \subset O_{f(z)}$. By the continuity of the extension operation r^* , there is a neighborhood $U_f \subset C_k(Z, Y)$ such that $r^*(U_f) \subset U_{f \circ r}$. Then for the neighborhood $Z \cap O_z$ of z in Z we get $U_f(Z \cap O_z) = U_{f \circ r}(Z \cap O_z) \subset U_{f \circ r}(O_z) \subset O_{f(z)}$, which means that the compact set $\mathcal{K} \subset C_k(Z, Y)$ is evenly continuous. Thus the space Z is Y -Ascoli.

(2) Let $\{X_i\}_{i \in I}$ be a family of Y -Ascoli spaces. To show that the topological sum $X = \bigoplus_{i \in I} X_i$ is Y -Ascoli, fix a compact set $\mathcal{K} \subset C_k(X, Y)$, a function $f \in \mathcal{K}$, a point $x \in X$ and a neighborhood $O_{f(x)}$ of $f(x)$ in Y . Find an index $i \in I$ such that $x \in X_i$. The continuity of the restriction operator $R_i : C_k(X, Y) \rightarrow C_k(X_i, Y)$, $R_i : f \mapsto f|_{X_i}$, implies that the subset $\mathcal{K}_i = R_i(\mathcal{K}) \subset C_k(X_i, Y)$ is compact. Since the space X_i is Y -Ascoli, there exist a neighborhood $U_{f|_{X_i}} \subset C_k(X_i, Y)$ of $f|_{X_i}$ and a neighborhood $O_x \subset X_i$ of x such that $U_{f|_{X_i}}(O_x) \subset O_{f(x)}$. By the continuity of the restriction operator $R_i : C_k(X, Y) \rightarrow C_k(X_i, Y)$, the set

$U_f = \{g \in C_k(X, Y) : g|_{X_i} \in U_{f|_{X_i}}\}$ is a neighborhood of f . Then $U_f(O_x) = U_{f|_{X_i}}(O_x) \subset O_{f(x)}$, which means that \mathcal{K} is evenly continuous and X is Y -Ascoli.

(3) For $j \in \kappa$, put $p_j : C_k(X, Y^\kappa) \rightarrow C_k(X, Y)$, $p_j : f \mapsto \pi_j \circ f$, where $\pi_j : Y^\kappa \rightarrow Y$ stands for the projection onto the j th coordinate. Then p_j is continuous. Let $\psi_\kappa : C_k(X, Y^\kappa) \times X \rightarrow Y^\kappa$ and $\psi : C_k(X, Y) \times X \rightarrow Y$ be the evaluation maps. Then $\pi_j \circ \psi_\kappa|_{\mathcal{K} \times X} = \psi \circ (p_j, \text{id}_X)|_{\mathcal{K} \times X}$ is continuous for every $j \in \kappa$ and every compact subset $\mathcal{K} \subset C_k(X, Y^\kappa)$, and therefore ψ_κ is continuous on $\mathcal{K} \times X$. Thus X is Y^κ -Ascoli.

(4) The last statement follows from the fact that for every subspace $Z \subset Y$ the function space $C_k(X, Z)$ is a subspace of $C_k(X, Y)$. \square

Since any (zero-dimensional) Tychonoff space embeds into some power \mathbb{R}^κ (respectively, 2^κ), Proposition 5.2 implies

Corollary 5.3. (1) *If X is an Ascoli space, then X is Y -Ascoli for every Tychonoff space Y .*
 (2) *If X is a 2-Ascoli space, then X is Y -Ascoli for every zero-dimensional T_1 -space Y .*

It turns out that the Y -Ascoli property of X is responsible for the continuity of the canonical map $\delta : X \rightarrow C_k(C_k(X, Y), Y)$.

Proposition 5.4. *For topological spaces X and Y , the canonical map $\delta : X \rightarrow C_k(C_k(X, Y), Y)$ is continuous if and only if X is Y -Ascoli.*

Proof. Assume that δ is continuous. To show that X is Y -Ascoli, we have to check that every compact subset \mathcal{K} of $C_k(X, Y)$ is evenly continuous. Fix $f \in \mathcal{K}$, $x \in X$ and an open neighborhood $O_{f(x)} \subset Y$ of $f(x)$. Using the regularity of Y , choose an open neighborhood $\tilde{O}_{f(x)}$ of $f(x)$ such that $\text{cl}_Y(\tilde{O}_{f(x)}) \subset O_{f(x)}$. Let $U_f := [\{x\}; \tilde{O}_{f(x)}] \cap \mathcal{K}$ and $\mathcal{C} := \text{cl}_{\mathcal{K}}(U_f \cap \mathcal{K})$. Then for every $g \in \mathcal{C}$ we have

$$\delta_x(g) = g(x) \in \text{cl}_Y(\tilde{O}_{f(x)}) \subset O_{f(x)},$$

and therefore $\delta_x \in [\mathcal{C}; O_{f(x)}]$. Since δ is continuous, there is a neighborhood O_x of x such that $\delta(O_x) \subset [\mathcal{C}; O_{f(x)}]$. So for every $y \in O_x$ and each $g \in U_f \subset \mathcal{C}$ we have $g(y) = \delta_y(g) \in O_{f(x)}$, which means that \mathcal{K} is evenly continuous.

Conversely, assume that X is Y -Ascoli. We have to show that δ is continuous at each point $x_0 \in X$. Fix a sub-basic neighborhood

$$[\mathcal{K}; V] \subset C_k(C_k(X, Y), Y)$$

of δ_{x_0} with $\mathcal{K} \subset C_k(X, Y)$ compact and $V \subset Y$ open. It follows from $\delta_{x_0} \in [\mathcal{K}; V]$ that $f(x_0) = \delta_{x_0}(f) \in V$ for every $f \in \mathcal{K}$. Since X is Y -Ascoli, for every $f \in \mathcal{K}$ there exist neighborhoods $U_f \subset \mathcal{K}$ of f and $O_f \subset X$ of x_0 such that $U_f(O_f) \subset V$. By the compactness of \mathcal{K} , there is a finite subset $\mathcal{F} \subset \mathcal{K}$ such that $\mathcal{K} = \bigcup_{f \in \mathcal{F}} U_f$. Consider the neighborhood $O_{x_0} := \bigcap_{f \in \mathcal{F}} O_f$. Then for every $x \in O_{x_0}$ and each $g \in \mathcal{K}$ we have

$$\delta_x(g) \in g(O_{x_0}) \subset \bigcup_{f \in \mathcal{F}} U_f(O_{x_0}) \subset \bigcup_{f \in \mathcal{F}} U_f(O_f) \subset V.$$

This means that $\delta_x \in [\mathcal{K}; V]$ and hence the canonical map δ is continuous at x_0 . \square

Next, we give some conditions on topological spaces X and Y guaranteeing that the canonical map $\delta : X \rightarrow C_k(X) \subset C_k(C_k(X, Y), Y)$ is injective or open. We shall say that a map $f : X \rightarrow Y$ between topological spaces is *open* if for any open set $U \subset X$ the image $f(U)$ is open in the subspace $f(X)$ of Y .

Definition 5.5. *Let X and Y be topological spaces. The topological space X is called*

- *Y -separated if for any distinct points $x, y \in X$ there is a continuous map $f : X \rightarrow Y$ such that $f(x) \neq f(y)$;*
- *Y -regular if for any point $x \in X$ and a neighborhood $O_x \subset X$ of x there is a continuous map $f : X \rightarrow Y$ such that $x \in f^{-1}(V) \subset O_x$ for some open set $V \subset Y$;*
- *Y -Tychonoff if X is Y -separated and Y -regular.*

Observe that each Y -Tychonoff space X embeds into some power Y^κ of the space Y . Observe also that a topological space X is Tychonoff if and only if X is \mathbb{R} -Tychonoff, and X is zero-dimensional if and only if it is 2-regular for the doubleton $2 = \{0, 1\}$ endowed with the discrete topology.

The following proposition can be easily derived from Definition 5.5 and Proposition 5.4.

Proposition 5.6. *For topological spaces X, Y the canonical map $\delta : X \rightarrow \delta(X) \subset C_k(C_k(X, Y), Y)$ is*

- (1) *injective if and only if X is Y -separated;*
- (2) *open if X is Y -regular;*
- (3) *open and injective if X is Y -Tychonoff;*
- (4) *continuous if and only if X is Y -Ascoli;*
- (5) *a topological embedding if X is Y -Ascoli and Y -Tychonoff.*

Remark 5.7. *Each connected space X is 2-Ascoli since $C_k(X, 2) = \{\mathbf{0}, \mathbf{1}\}$. It follows that the canonical map $\delta : X \rightarrow C_k(C_k(X, 2), 2)$ is constant. Hence δ is continuous and open, but it is not injective if $|X| > 1$.*

Since the class of k -spaces is (properly) contained in the class of Ascoli spaces, the following corollary generalizes Theorem 2.3.6 of [17].

Corollary 5.8. *For every Ascoli Tychonoff space X the canonical map $\delta : X \rightarrow C_k(C_k(X))$ is a topological embedding.*

Our interest to the study of Ascoli spaces can be explained by the following theorem which gives a “lower bound” on the C_k -stable closure $\mathbf{C}_k[\mathcal{X}]$ of any non-empty class \mathcal{X} of topological spaces. The second part of this theorem is Theorem 1.8.

Theorem 5.9. *For any non-empty class \mathcal{X} of topological spaces the class $\mathbf{C}_k[\mathcal{X}]$ contains all zero-dimensional 2-Ascoli \aleph_0 -spaces. If $\mathbb{I} \in \mathcal{X}$, then the class $\mathbf{C}_k[\mathcal{X}]$ contains all Ascoli \aleph_0 -spaces.*

Proof. By Proposition 2.1, the class \mathcal{X} contains all zero-dimensional separable metrizable spaces. Now take any zero-dimensional 2-Ascoli \aleph_0 -space X and consider the canonical map $\delta : X \rightarrow C_k(C_k(X, 2), 2)$. By Proposition 5.6, δ is an embedding. Applying Proposition 2.1(3) twice, we obtain that $C_k(C_k(X, 2), 2) \in \mathbf{C}_k[\mathcal{X}]$. Thus also $X \in \mathbf{C}_k[\mathcal{X}]$.

Now assume that $\mathbb{I} \in \mathcal{X}$. Given any Ascoli \aleph_0 -space X , we apply Proposition 5.6 to conclude that the canonical map $\delta : X \rightarrow C_k(C_k(X, \mathbb{I}), \mathbb{I})$ is a topological embedding. By Proposition 2.1, the double function space $C_k(C_k(X, \mathbb{I}), \mathbb{I})$ and hence also its subspace X belong to the class $\mathbf{C}_k[\mathcal{X}]$. \square

Below we propose a characterization of Ascoli spaces which will be applied for constructing an Ascoli space X which is not a $k_{\mathbb{R}}$ -space in Example 6.7.

Proposition 5.10. *A topological space X is Ascoli if and only if each point $x \in X$ is contained in a dense Ascoli subspace of X .*

Proof. The “only if” part is trivial. To prove the “if” part, assume that each point $x \in X$ is contained in a dense Ascoli subspace of X . Choose any compact set $\mathcal{K} \subset C_k(X)$, function $f \in \mathcal{K}$, point $x \in X$, and neighborhood $O_{f(x)} \subset \mathbb{R}$ of $f(x)$. By the regularity of \mathbb{R} , there is a neighborhood $W_{f(x)} \subset \mathbb{R}$ of $f(x)$ such that $\text{cl}_{\mathbb{R}}(W_{f(x)}) \subset O_{f(x)}$. By our assumption, the point x is contained in a dense Ascoli subspace $Z \subset X$. The density of Z in X implies that the restriction operator

$$\zeta : C_k(X) \rightarrow C_k(Z), \quad \zeta : g \mapsto g|_Z,$$

is injective. Since the space Z is Ascoli, for the compact subset $\zeta(\mathcal{K}) \subset C_k(Z)$, the function $h := f|_Z$ and the neighborhood $W_{f(x)}$ of $h(x) = f(x)$ there are neighborhoods $U_h \subset \zeta(\mathcal{K})$ of h and $W_x \subset Z$ of x such that $U_h(W_x) \subset W_{f(x)}$. It follows that $U_f := \{g \in \mathcal{K} : g|_Z \in U_h\}$ is a neighborhood of f in \mathcal{K} and the closure $\overline{W_x}$ of W_x in X is a (closed) neighborhood of x in X such that $U_f(\overline{W_x}) \subset \overline{W_{f(x)}} \subset O_{f(x)}$. Thus \mathcal{K} is evenly continuous. \square

Now we prove that the classes of Ascoli and $k_{\mathbb{R}}$ -spaces are hereditary with respect to taking closed subspaces in stratifiable spaces. We recall (see [15, §5]) that a regular topological space X is *stratifiable* if there is a function G which assigns to every $n \in \omega$ and each closed set $F \subset X$ an open neighborhood $G(n, F) \subset X$ of F such that $F = \bigcap_{n \in \omega} \overline{G(n, F)}$ and $G(n, F) \subset G(n, F')$ for any $n \in \omega$ and closed sets $F \subset F' \subset X$. Reznichenko in [23] proved that for a separable metrizable space X the function space $C_k(X)$ is stratifiable if and only if the space X is Polish.

Proposition 5.11. *Let A be a closed subspace of a stratifiable space X . Then*

- (1) if X is a $k_{\mathbb{R}}$ -space, then A is a $k_{\mathbb{R}}$ -space;
- (2) if X is Ascoli, then A is Ascoli;
- (3) if X is 2-Ascoli and $|X \setminus A| < \mathfrak{c}$, then A is 2-Ascoli.

Proof. To prove the proposition we need the following construction due to Borges (see Section 4 and the proof of Theorem 4.3 [6]). Let $[A]^{<\omega}$ be the set of non-empty finite subsets of A and $\mathbf{P}_\omega(A)$ be the space of probability measures with finite support on A . The space $\mathbf{P}_\omega(A)$ is considered with the weak-star topology inherited from the dual space $C_b^*(A)$ of the Banach space $C_b(A)$ of all bounded continuous real-valued functions on A endowed with the sup-norm. The weak-star topology on the dual Banach space $C_b^*(A)$ is inherited from the Tychonoff product topology on $\mathbb{R}^{C_b(A)}$. A map $u : X \rightarrow [A]^{<\omega}$ is called *upper semicontinuous* if for any open set $U \subset A$ the set $\{x \in X : u(x) \subset U\}$ is open in X . It was shown in (the proof of Theorem 4.3 of) [6] that there exist an upper semicontinuous map $u : X \rightarrow [A]^{<\omega}$ with $u(a) = \{a\}$ for all $a \in A$, and a continuous map $\mu : X \rightarrow \mathbf{P}_\omega(A)$ such that for each point $x \in X$ the finite set $u(x)$ has measure 1 with respect to the measure $\mu(x)$ which will be denoted by μ_x . Since for any point $a \in A$ the set $u(a)$ coincides with the singleton $\{a\}$, the measure μ_a coincides with the Dirac measure δ_a concentrated at a . Then the map $\mu : X \rightarrow \mathbf{P}_\omega(X)$ induces a continuous linear operator $e : C_k(A) \rightarrow C_k(X)$ assigning to each function $f \in C_k(A)$ the function $e(f) \in C_k(X)$, $e(f) : x \mapsto \mu_x(f)$ (the value $e(f)(x) = \mu_x(f)$ is well-defined as the measure μ_x has finite support). Observe that for every point $a \in A$ we get $e(f)(a) = \mu_a(f) = \delta_a(f) = f(a)$, so e is an extension operator.

(1) Assume that the space X is a $k_{\mathbb{R}}$ -space. To show that A is a $k_{\mathbb{R}}$ -space, take any function $f : A \rightarrow \mathbb{R}$ such that for every compact subset $K \subset A$ the restriction $f|_K$ is continuous. Consider the function $\tilde{f} : X \rightarrow \mathbb{R}$ defined by $\tilde{f}(x) := e(f)(x) = \mu_x(f)$ for $x \in X$. We show that for every compact subset $K \subset X$ the restriction $\tilde{f}|_K$ is continuous. The upper semicontinuity of the map u and the compactness of K imply the compactness of the set

$$u[K] := \bigcup_{x \in K} u(x) \subset A.$$

By our assumption, the restriction $f|_{u[K]}$ is continuous. Then the continuity of the map $\mu|_K : K \rightarrow \mathbf{P}_\omega(u[K])$ guarantees the continuity of the function $\tilde{f}|_K$ (as $\tilde{f}(x) = \mu_x(f|_{u[K]})$ for any $x \in K$). Taking into account that X is a $k_{\mathbb{R}}$ -space, we conclude that the function $\tilde{f} : X \rightarrow \mathbb{R}$ is continuous and so is its restriction $f = \tilde{f}|_A$.

(2) Now assume that X is an Ascoli space. To show that the space A is Ascoli, fix any compact subset $\mathcal{K} \subset C_k(A)$. Given a function $f \in \mathcal{K}$, a point $a \in A$ and a neighborhood $O_{f(a)} \subset \mathbb{R}$ of $f(a)$, we need to find neighborhoods $U_f \subset \mathcal{K}$ of f and $O_a \subset A$ of a such that $U_f(O_a) \subset O_{f(a)}$. As the extension operator $e : C_k(A) \rightarrow C_k(X)$ is continuous, the set $\tilde{\mathcal{K}} := e(\mathcal{K}) \subset C_k(X)$ is compact. Since the space X is Ascoli, for the function $\tilde{f} = e(f) \in \tilde{\mathcal{K}}$ there are neighborhoods $U_{\tilde{f}} \subset \tilde{\mathcal{K}}$ of \tilde{f} and $\tilde{O}_a \subset X$ of a such that $U_{\tilde{f}}(\tilde{O}_a) \subset O_{\tilde{f}(a)}$. By the continuity of the operator e , there is a neighborhood $U_f \subset \mathcal{K}$ of f such that $e(U_f) \subset U_{\tilde{f}}$. Then for the neighborhood $O_a = \tilde{O}_a \cap A$ of a in A , we get $U_f(O_a) \subset U_{\tilde{f}}(\tilde{O}_a \cap A) \subset O_{\tilde{f}(a)}$, which means that the compact set \mathcal{K} is evenly continuous.

(3) Assume that X is 2-Ascoli and $|X \setminus A| < \mathfrak{c}$. Consider the map $\mu : X \rightarrow \mathbf{P}_\omega(A)$, and observe that for every $x \in X$ the set

$$Y_x = \{\mu_x(B) : B \subset X\}$$

of all possible values of the measure $\mu_x := \mu(x)$ is finite. Since $|X \setminus A| < \mathfrak{c}$, the set $Y = \bigcup_{x \in X \setminus A} Y_x \subset [0, 1]$ has cardinality $< \mathfrak{c}$. By (the proof of) Corollary 6.2.8 in [7], the regular space Y is zero-dimensional. Now Corollary 5.3 implies that the 2-Ascoli space X is Y -Ascoli.

To show that the space A is 2-Ascoli, fix any compact subset $\mathcal{K} \subset C_k(A, 2)$. Given a function $f \in \mathcal{K}$, a point $a \in A$ and a neighborhood $O_{f(a)} \subset 2 = \{0, 1\}$ of $f(a)$, we need to find neighborhoods $U_f \subset \mathcal{K}$ of f and $O_a \subset A$ of a such that $U_f(O_a) \subset O_{f(a)}$. Take any neighborhood $\tilde{O}_{f(a)} \subset \mathbb{R}$ of $f(a)$ such that $\tilde{O}_{f(a)} \cap \{0, 1\} = O_{f(a)}$. Consider the compact set $\tilde{\mathcal{K}} := e(\mathcal{K}) \subset C_k(X)$ and the function $\tilde{f} = e(f) \in \tilde{\mathcal{K}} \subset C_k(X)$. The definition of the extension operation e implies that $\tilde{\mathcal{K}} \subset C_k(X, Y) \subset C_k(X, \mathbb{R})$. Since the space X is Y -Ascoli, there are neighborhoods $U_{\tilde{f}} \subset \tilde{\mathcal{K}}$ of \tilde{f} and $\tilde{O}_a \subset X$ of a such that $U_{\tilde{f}}(\tilde{O}_a) \subset \tilde{O}_{\tilde{f}(a)}$. By the continuity of the operator e , there is a neighborhood $U_f \subset \mathcal{K}$ of f such that $e(U_f) \subset U_{\tilde{f}}$. Then for the neighborhood $O_a = \tilde{O}_a \cap A$ of a in A , we get $U_f(O_a) \subset \{0, 1\} \cap U_{\tilde{f}}(\tilde{O}_a \cap A) \subset \{0, 1\} \cap \tilde{O}_{\tilde{f}(a)} = O_{f(a)}$, which means that the compact set \mathcal{K} is evenly continuous, and A is 2-Ascoli. \square

The following proposition will be used also in the last section. Recall that a topological space X is called *scattered* if each non-empty subspace of X contains an isolated point.

Proposition 5.12. *A zero-dimensional 2-Ascoli space X is sequential if it satisfies one of the following conditions:*

- (1) *each compact subset of X is finite (in this case X is discrete);*
- (2) *X is stratifiable, scattered, and has cardinality $|X| < \mathfrak{c}$.*

Proof. (1) Assume that the 2-Ascoli space X does not contain infinite compact subsets. We shall prove that X is discrete (and hence sequential). Assuming that X is not discrete, fix a non-isolated point $x \in X$. By Zorn's Lemma, there exists a maximal disjoint family \mathcal{U} of non-empty closed-and-open subsets of X which do not contain the point x . By the maximality of \mathcal{U} , the point x belongs to the closure of the union $\bigcup \mathcal{U}$ in X . Since all compact subsets of X are finite, the set of characteristic functions $K = \{\chi_\emptyset\} \cup \{\chi_U : U \in \mathcal{U}\} \subset C_k(X, 2)$ is compact with a unique non-isolated point χ_\emptyset . Since X is 2-Ascoli, the set \mathcal{K} is evenly continuous. Consequently, we can find a neighborhood $U_{\chi_\emptyset} \subset \mathcal{K}$ of the constant zero function $\chi_\emptyset : X \rightarrow 2$ and a neighborhood $O_x \subset X$ of the point x such that $U_{\chi_\emptyset}(O_x) \subset \{\emptyset\}$.

As χ_\emptyset is a unique non-isolated point of the compact set \mathcal{K} , the set $\mathcal{K} \setminus U_{\chi_\emptyset}$ is finite. Since the neighborhood O_x meets infinitely many sets $U \in \mathcal{U}$, we can find a set $U \in \mathcal{U}$ such that $O_x \cap U \neq \emptyset$ and $\chi_U \in U_{\chi_\emptyset}$. Then for any point $u \in O_x \cap U$ we get $1 = \chi_U(u) \in U_{\chi_\emptyset}(O_x) \subset \{\emptyset\}$, a contradiction. Thus the space X is discrete (and sequential).

(2) Assume that X is a 2-Ascoli stratifiable scattered space of cardinality $|X| < \mathfrak{c}$. We need to prove that the space X is sequential. Since all compact subsets of the stratifiable space X are metrizable (see [15, 4.7 and 5.9]), it suffices to show that X is a k -space. Assuming the opposite, we can find a non-closed subset $A \subset X$ such that for each compact subset $K \subset X$ the intersection $K \cap A$ is compact. By Proposition 5.11, the closed subspace \bar{A} of the stratifiable 2-Ascoli space X is 2-Ascoli. Replacing the space X by the closure \bar{A} of A in X , we can assume that the set A is dense in X .

For a subset $B \subset X$ let $B^{(1)}$ denote the set of all non-isolated points of B . Let $X^{(0)} = X$, and for every ordinal $\alpha > 0$ put $X^{(\alpha)} = \bigcap_{\beta < \alpha} (X^{(\beta)})^{(1)}$. Since X is scattered, for some ordinal α the set $X^{(\alpha)}$ is empty. So for each point $x \in X$ we can assign the unique ordinal $h(x)$ (called the *scattered height* of x) such that $x \in X^{(h(x))} \setminus X^{(h(x)+1)}$. Consider the ordinal $\alpha = \min\{h(x) : x \in X \setminus A\}$ and choose a point $a \in X \setminus A$ with $h(a) = \alpha$. Since a is an isolated point of the set $X^{(\alpha)} \setminus X^{(\alpha+1)}$ and X is zero-dimensional, we can find a closed-and-open neighborhood $O_a \subset X$ of a such that $O_a \cap X^{(\alpha)} = \{a\}$. This means that each point $x \in O_a \setminus \{a\}$ has scattered height $h(x) < \alpha$. The definition of the ordinal α implies that $O_a \setminus A = \{a\}$ and hence $O_a \cap A = O_a \setminus \{a\}$.

The space $Z := O_a \setminus \{a\}$ is stratifiable, and hence paracompact [15, 5.7]. By (the proof of) Corollary 6.2.8 of [7], the space Z of cardinality $|Z| < \mathfrak{c}$ is strongly zero-dimensional and hence has covering dimension $\dim(Z) = 0$. Since Z is paracompact and has covering dimension zero we can apply Dowker's Theorem [7, 7.2.4] and conclude that the open cover

$$\mathcal{W} = \{Z \setminus U : U \text{ is a clopen neighborhood of } a \text{ in } O_a\}$$

of Z has a disjoint open refinement \mathcal{V} covering Z . Observe that each element $V \in \mathcal{V}$ is closed in O_a since $V = (O_a \setminus U) \setminus \bigcup\{V' \in \mathcal{V} : V' \neq V\}$ for some clopen subset U in O_a . Since a is an accumulation point of $O_a \setminus \{a\}$, each neighborhood $U_a \subset X$ of a meets infinitely many (clopen) sets $V \in \mathcal{V}$.

We claim that the set of characteristic functions

$$\mathcal{K} = \{\chi_\emptyset\} \cup \{\chi_V : V \in \mathcal{V}\}$$

is compact in $C_k(X, 2)$ and has a unique non-isolated point χ_\emptyset . It suffices to check that any neighborhood $O_{\chi_\emptyset} \subset C_k(X, 2)$ of χ_\emptyset contains all but finitely many functions $f \in \mathcal{K}$. Without loss of generality we can assume that the neighborhood O_{χ_\emptyset} is of the basic form

$$O_{\chi_\emptyset} = \{f \in C_k(X, 2) : f(K) \subset \{0\}\},$$

for some compact set $K \subset X$ containing the point a . By the choice of the set A , the intersection $A \cap K$ is compact and so is its closed subset $(O_a \cap A) \cap K = (O_a \setminus \{a\}) \cap K$. This means that a is an isolated point of the compact space K . Since \mathcal{V} is a disjoint open cover of $O_a \setminus \{a\}$, the compactness of $(O_a \setminus \{a\}) \cap K$ guarantees

that K meets only finitely many sets $V \in \mathcal{V}$. This implies that the neighborhood O_{χ_\emptyset} contains all but finitely many characteristic functions χ_V , $V \in \mathcal{V}$. So, $\mathcal{K} \subset C_k(X, 2)$ is a compact set with the unique non-isolated point χ_\emptyset .

Since the space X is 2-Ascoli, the compact set \mathcal{K} is evenly continuous. This allows us to find a neighborhood $U_{\chi_\emptyset} \subset \mathcal{K}$ of the constant zero function χ_\emptyset and a neighborhood $W_a \subset X$ of the point a such that $U_{\chi_\emptyset}(W_a) \subset \{0\}$. Since χ_\emptyset is a unique non-isolated point of the compact set \mathcal{K} the set

$$\mathcal{V}' = \{V \in \mathcal{V} : \chi_V \in U_{\chi_\emptyset}\}$$

has finite complement $\mathcal{V} \setminus \mathcal{V}'$. As each neighborhood of a meets infinitely many sets $V \in \mathcal{V}$, we can find a set $V \in \mathcal{V}'$ such that the intersection $W_a \cap V$ contains some point v . Then $1 = \chi_V(v) \in U_{\chi_\emptyset}(W_a) \subset \{0\}$, a contradiction. Thus the space X is sequential. \square

Now we present an example of two Fréchet-Urysohn stratifiable \aleph_0 -spaces X and Y whose product $X \times Y$ is 2-Ascoli but not Ascoli. The space X is the following σ -compact subspace of the real plane

$$X := \bigcup_{n \in \omega \setminus \{0\}} \{(t, t/n) : t \in [0, 1]\} \subset \mathbb{R}^2,$$

which is called the *connected metric fan*. The space Y is the space X endowed with the strongest topology inducing the Euclidean topology on each arc

$$I_n = \{(t, t/n) : t \in [0, 1]\}, \quad n > 0.$$

The space Y is called the *connected Fréchet-Urysohn fan* and is a (non-metrizable) Fréchet-Urysohn k_ω -space.

The following proposition shows that the class of Ascoli spaces is not productive and the class of 2-Ascoli spaces is neither productive nor closed hereditary.

Proposition 5.13. *The spaces X and Y have the following properties:*

- (1) X is separable and metrizable, while Y is a Fréchet-Urysohn stratifiable \aleph_0 -space;
- (2) the spaces X and Y are Ascoli;
- (3) the product $X \times Y$ is 2-Ascoli but is not Ascoli;
- (4) the spaces X and Y contain closed countable scattered subspaces $X_0 \subset X$ and $Y_0 \subset Y$ whose product $X_0 \times Y_0$ is not 2-Ascoli.

Proof. The metrizability and separability of X is clear. It follows that the space Y is the image of the metrizable separable space $\bigoplus_{n>0} I_n$ under a closed compact-covering map. This implies that Y is a stratifiable \aleph_0 -space, see Theorem 5.5 in [15]. Then the product $X \times Y$ is a stratifiable \aleph_0 -space (by the productivity of the classes of stratifiable and \aleph_0 -spaces, see [15, 5.10 and 11.2]). The spaces X and Y are Ascoli spaces (being k -spaces). The connectedness of the space $X \times Y$ implies that this space is 2-Ascoli.

In the spaces X and Y consider the countable subspaces

$$X_0 = \{x_\infty\} \cup \{x_{n,m} : n, m > 0\} \subset X \quad \text{and} \quad Y_0 = \{y_\infty\} \cup \{y_{n,m} : n, m > 0\} \subset Y,$$

where

$$x_\infty = (0, 0) = y_\infty, \quad x_{n,m} = \left(\frac{1}{n}, \frac{1}{nm}\right) \quad \text{and} \quad y_{n,m} = \left(\frac{1}{m}, \frac{1}{mn}\right), \quad \forall n, m > 0.$$

It follows that for every $n > 0$ the set $\{x_{n,m}\}_{m>0}$ is closed and discrete in X , while the sequence $\{y_{n,m}\}_{m>0} \subset I_n$ converges to y_∞ in Y . It is known (see, e.g. [1]) that the space $X_0 \times Y_0$ is not sequential. By Proposition 5.12(2), the countable scattered space $X_0 \times Y_0$ is not 2-Ascoli and hence not Ascoli. Since $X_0 \times Y_0$ is a closed subspace of the stratifiable space $X \times Y$, we can apply Proposition 5.11 to conclude that the space $X \times Y$ is not Ascoli. \square

Problem 5.14. *Is each zero-dimensional 2-Ascoli space Ascoli?*

6. EXAMPLES AND OPEN QUESTIONS

In this section we provide examples which show that the implications in the diagram presented in the introduction cannot be reversed and pose several open questions.

Example 6.1. *There exists an Ascoli \aleph_0 -space which is a $k_{\mathbb{R}}$ -space but is not a k -space.*

Proof. In [20] Michael constructed an \aleph_0 -space X , which is a $k_{\mathbb{R}}$ -space but is not a k -space (and hence is not sequential). By [21], the space X is Ascoli. \square

Example 6.2. *The class $C_k[\mathfrak{M}_0]$ contains a countable topological group Δ such that*

- Δ is not discrete but all compact subsets of Δ are finite;
- Δ embeds into the product $F \times G$ of a countable k_ω -group F and a metrizable group G ;
- Δ is stratifiable;
- Δ is not 2-Ascoli and hence is not Ascoli.

Proof. Let F be the free abelian topological group over the convergent sequence $\{0\} \cup \{\frac{1}{n} : n > 0\} \subset \mathbb{R}$. It is well-known that F is a countable k_ω -space and hence F is a sequential stratifiable \aleph_0 -space (for the stratifiability of F , see [15, 5.5]). By Theorem 1.8, the topological space F belongs to the class $C_k(\mathfrak{M}_0)$ (recall that any sequential space is Ascoli).

Denote by G the free abelian group F , endowed with the metrizable group topology τ whose neighborhood base at zero consists of the subgroups $2^k F$, $k \in \omega$. Being metrizable and separable, the topological group G is a $C_k[\mathfrak{M}_0]$ -space. By [2], the diagonal subgroup $\Delta = \{(x, y) \in F \times G : x = y\}$ is not discrete, and by [10] every compact subset of Δ is finite. It follows that the space Δ is not a k -space. Applying Proposition 5.12(1), we conclude that Δ is not 2-Ascoli (and hence not Ascoli). Since both spaces F and G are stratifiable, so are their product $F \times G$ and the subspace $\Delta \subset F \times G$. \square

Remark 6.3. *The stratifiable space $X \times Y$ from Proposition 5.13 also belongs to the class $C_k[\mathfrak{M}_0]$ (by productivity of $C_k[\mathfrak{M}_0]$) but fails to be Ascoli.*

Example 6.4. *There exists a \mathfrak{P}_0 -space, which is not a $C_k[\mathfrak{M}]$ -space.*

Proof. Let X be a separable metrizable space containing a topological copy of the Cantor cube 2^ω . Since the Cantor cube is homeomorphic to its own square, the space X contains an uncountable family \mathcal{C} of pairwise disjoint topological copies of 2^ω . Enlarge the family \mathcal{C} to the smallest discretely-complete ideal \mathcal{I} of compact subsets of X . Hence any element of \mathcal{I} is contained in the union of a finite subfamily of \mathcal{C} and a discrete subset of X . So, the union of any infinite subfamily of \mathcal{C} does not belong to \mathcal{I} , and therefore \mathcal{I} fails to be a $P_\omega^{\omega^1}$ -ideal. Then by Theorem 4.5, the function space $C_{\mathcal{I}}(X)$ fails to be a $P_\omega^{\omega^1}$ -space and hence cannot be a $C_k[\mathfrak{M}]$ -space. On the other hand, Theorem 3.2 guarantees that the function space $C_{\mathcal{I}}(X)$ is a \mathfrak{P}_0 -space. \square

Remark 6.5. *In [12] it is shown that the precompact group \mathbb{Z}^\sharp of integers endowed with the Bohr topology is an \aleph_0 -space but fails to be a \mathfrak{P} -space.*

Problem 6.6. *Is there an Ascoli \aleph -space which is not a $C_k[\mathfrak{M}]$ -space?*

Example 6.7. *Let λ and $\kappa > \lambda$ be infinite cardinals and Y be a Tychonoff first countable space containing more than one point. Then the subspace*

$$X = \bigcup_{y \in Y} \{f \in Y^\kappa : |f^{-1}(Y \setminus \{y\})| < \lambda\}$$

of Y^κ is Ascoli but fails to be a $k_{\mathbb{R}}$ -space. If Y is a topological group (or a linear topological space), then so is the space X .

Proof. To see that X is Ascoli, it suffices to check that each element $f \in X$ is contained in a dense Ascoli subspace of X . By [7, 3.10.D]), the σ -product $\sigma(f) = \{g \in Y^\kappa : |\{x \in \kappa : f(x) \neq g(x)\}| < \omega\} \subset Y^\kappa$ is Fréchet–Urysohn and hence Ascoli according to the Ascoli Theorem [7, 3.4.20]. Clearly, $\sigma(f)$ is a dense subset of X , and therefore X is Ascoli by Proposition 5.10.

To show that the space X is not a $k_{\mathbb{R}}$ -space, consider the map $\lim : X \rightarrow Y$ assigning to each function $f \in X$ the unique point $y \in Y$ such that the set $\text{supp}(f) := f^{-1}(Y \setminus \{y\})$ has cardinality $|\text{supp}(f)| < \lambda$. It is clear that the map \lim is discontinuous. We claim that for every compact subset $K \subset X$ the restriction $\lim|_K$ is continuous, and therefore X is not a $k_{\mathbb{R}}$ -space.

It suffices to prove that, for each closed subset $B \subset Y$, the set $D := \{f \in K : \lim f \in B\}$ is closed in K . Suppose for a contradiction that the set D is not closed in K and has an accumulation point $f \in K \setminus D$. Then $y = \lim f \notin B$. Take any subset $N \subset \kappa \setminus \text{supp}(f)$ of cardinality $|N| = \lambda$. It follows from $N \cap \text{supp}(f) = \emptyset$ that $f(N) = \{y\}$. Let $\mathcal{F}(N)$ be the family of all finite subsets of N , partially ordered by the inclusion relation.

By the regularity of Y , the point y has a closed neighborhood $\bar{O}_y \subset Y$, disjoint with the closed set B . Since f is an accumulation point of the set D , for every finite subset $F \in \mathcal{F}(N)$ we can choose a function $f_F \in D \subset K$

such that $f_F(F) \subset \bar{O}_y$. By the compactness of K , the net $(f_F)_{F \in \mathcal{F}(N)}$ has a limit point $f_\infty \in K$. So for any neighborhood $O(f_\infty) \subset X \subset Y^\kappa$ of f_∞ and any $F \in \mathcal{F}(N)$ there is an element $E \in \mathcal{F}(N)$ such that $F \subset E$ and $f_E \in O(f_\infty)$, and therefore $f_\infty(N) \subset \bar{O}_y$.

Now consider the set $S := \bigcup_{F \in \mathcal{F}(N)} \text{supp}(f_F)$ and observe that $|S| \leq |N| < \kappa$. Since the set $\{f_F\}_{F \in \mathcal{F}(N)}$ is contained in the closed subset $\{g \in Y^\kappa : g(\kappa \setminus S) \subset B\}$ of Y^κ the limit point f_∞ has the property $f_\infty(X \setminus S) \subset B$. Since the sets N and $X \setminus S$ have cardinality $\geq \lambda$ and $f_\infty(N) \cap f_\infty(X \setminus S) \subset \bar{O}_y \cap B = \emptyset$, the function f_∞ does not belong to X and hence $f_\infty \notin K$. This contradiction completes the proof of the continuity of the restriction $\lim|_K$. \square

The Ascoli space constructed in Example 6.7 is not cosmic and hence not an \aleph_0 -space.

Problem 6.8. *Is there an Ascoli space X which is cosmic (or an \aleph_0 -space) but fails to be a $k_{\mathbb{R}}$ -space?*

In Theorem 1.3 we characterized $C_k[\mathfrak{M}_0]$ -spaces via embeddings into function spaces between separable metrizable spaces.

Problem 6.9. *Give an inner characterization of $C_k[\mathfrak{M}_0]$ -spaces (desirably, in terms of special networks).*

By definition, the class $\mathbf{C}_k[\mathfrak{M}]$ is closed under countable topological sums.

Problem 6.10. *Is the class $\mathbf{C}_k[\mathfrak{M}]$ closed under taking arbitrary topological sums?*

This problem is related to another open problem. Let \mathcal{X} be a class of topological spaces. A topological space $U \in \mathcal{X}$ is called *universal* in \mathcal{X} if U contains a topological copy of any space $X \in \mathcal{X}$. Since any discrete space D belongs to $\mathbf{C}_k[\mathfrak{M}]$, the class $\mathbf{C}_k[\mathfrak{M}]$ does not have universal spaces.

Problem 6.11. *Is there a universal space U in the class $\mathbf{C}_k[\mathfrak{M}_0]$?*

The next proposition describes a relation between Problems 6.10 and 6.11.

Proposition 6.12. *If the class $\mathbf{C}_k[\mathfrak{M}]$ is closed under taking topological sums, then the class $\mathbf{C}_k[\mathfrak{M}_0]$ contains a universal space.*

Proof. By assumption, for the topological sum $\bigoplus_{X \subset \mathbb{R}^\omega} C_k(X)$ there is a topological embedding

$$e : \bigoplus_{X \subset \mathbb{R}^\omega} C_k(X) \hookrightarrow C_k(Z, Y)$$

for some spaces $Z \in \mathfrak{M}_0$ and $Y \in \mathfrak{M}$. For every subspace $X \subset \mathbb{R}^\omega$, the function space $C_k(X)$ is Lindelöf and so is its topological copy $e(C_k(X))$ in $C_k(Z, Y)$. Repeating the argument from the proof of Theorem 2.3, we can find a separable subspace $Y_X \subset Y$ such that $e(C_k(X)) \subset C_k(Z, Y_X)$. The space Y_X , being separable and metrizable, embeds into the countable product \mathbb{R}^ω of the real line. Then the function space $C_k(Z, Y_X)$ embeds into $C_k(Z, \mathbb{R}^\omega)$, which is homeomorphic to $C_k(Z \times \omega)$. This means that the space $C_k(Z \times \omega)$ is universal in the class $\mathbf{C}_k[\mathfrak{M}_0]$. \square

We expect that the answer to Problems 6.10 and 6.11 are negative. Let us observe the following two facts.

Proposition 6.13. *Let X and Y be two separable metrizable spaces such that the function space $C_k(X)$ embeds into $C_k(Y)$.*

- (1) *If Y is locally compact, then so is X .*
- (2) *If Y is Polish, then so is the space X .*

Proof. (1) If Y is a locally compact separable metrizable space, then the function space $C_k(Y)$ is metrizable and so is the space $C_k(X)$. By [17, 4.4.2] and [7, 3.4.E], the space X is locally compact.

(2) If Y is Polish, then by [8], the function space $C_k(Y)$ has a \mathfrak{G} -base at zero-function $\mathbf{0} \in C_k(Y)$. The latter means that each point of $C_k(Y)$ has a neighborhood base $(U_\alpha)_{\alpha \in \omega^\omega}$ indexed by functions $\alpha : \omega \rightarrow \omega$ such that $U_\alpha \subset U_\beta$ for any functions $\beta \leq \alpha$ in ω^ω . The function space $C_k(X)$, being a subspace of $C_k(Y)$ also has a \mathfrak{G} -base at each point. In this case Corollary 3 of [8] implies that the space X is Polish.

The same conclusion could be derived from the Reznichenko's characterization [23] of Polish spaces as separable metrizable spaces with stratifiable function spaces $C_k(X)$. Indeed, if Y is Polish, then by [23], the function space $C_k(Y)$ is stratifiable and so is its subspace $C_k(X)$. Applying the Reznichenko's characterization once again, we conclude that the separable metrizable space X is Polish. \square

Proposition 6.13 suggests the following problem (or rather, a program of research).

Problem 6.14. *Let X, Y be two separable metrizable spaces such that the function space $C_k(X)$ embeds into $C_k(Y)$. Which topological properties of Y are inherited by X ? In particular, if Y belongs to certain Borel or projective class, does then X belong to the same Borel or projective class?*

Remark 6.15. *In [4, 11, 14, 22] Ascoli spaces were detected among function spaces, locally convex spaces, and some spaces appearing in Topological Algebra.*

Acknowledgments The authors are deeply indebted to Professor R. Pol for fruitful discussion on the Ascoli property in function spaces. We would like to thank the referee for valuable remarks and suggestions.

REFERENCES

- [1] Banach T.: On topological groups containing a Frechet-Urysohn fan. *Mat. Stud.* **9**, 149–154 (1998)
- [2] Banach T.: Topologies on groups determined by sequences: answers to several questions of I. Protasov and E. Zelenyuk. *Mat. Stud.* (2) **15**, 145–150 (2001)
- [3] Banach T.: \mathfrak{P}_0 -spaces. *Topology Appl.* (to appear) (<http://arxiv.org/abs/1311.1468>)
- [4] Banach T.: Generalizations of k -spaces and their applications in general topology, function spaces, and Banach space theory. preprint.
- [5] Banach T., Gabrielyan S.: The C_p -stable closure of the class of separable metrizable spaces. preprint (<http://arxiv.org/abs/1412.2240>)
- [6] Borges C.: On stratifiable spaces. *Pacific J. Math.* **17**, 1–16 (1966)
- [7] Engelking R.: General topology. Heldermann Verlag, Berlin, (1989)
- [8] Ferrando J. C., Kąkol J.: On precompact sets in spaces $C_c(X)$. *Georgian Math. J.* **20**, 247–254 (2013)
- [9] Foged L.: Characterizations of \aleph -spaces. *Pacific J. Math.* **110**, 59–63 (1984)
- [10] Gabrielyan S.: Topologies on groups determined by sets of convergent sequences. *J. Pure Appl. Algebra* **217**, 786–802 (2013)
- [11] Gabrielyan S.: Topological properties of function spaces $C_k(X, 2)$ over zero-dimensional metric spaces X . submitted (<http://arxiv.org/abs/1504.04198>)
- [12] Gabrielyan S., Kąkol J.: On \mathfrak{P} -spaces and related concepts. submitted (<http://arxiv.org/abs/1412.1494>)
- [13] Gabrielyan S., Kąkol J., Kubiś W., Marciszewski W.: Networks for the weak topology of Banach and Fréchet spaces. submitted (<http://arxiv.org/abs/1412.1748>)
- [14] Gabrielyan S., Kąkol J., Plebanek G.: The Ascoli property for function spaces and the weak topology of Banach and Fréchet spaces. submitted (<http://arxiv.org/abs/1504.04202>).
- [15] Gruenhage G.: Generalized metric spaces. *Handbook of set-theoretic topology*, 423–501, North-Holland, Amsterdam, (1984)
- [16] Kechris A.: Classical Descriptive Set Theory. Springer-Verlag, New York, (1995)
- [17] McCoy R. A., Ntantu I.: Topological Properties of Spaces of Continuous Functions. *Lecture Notes in Math.* **1315** (1988)
- [18] O’Meara P.: On paracompactness in function spaces with the compact-open topology. *Proc. Amer. Math. Soc.* **29** 183–189 (1971)
- [19] Michael E.: \aleph_0 -spaces. *J. Math. Mech.* **15**, 983–1002 (1966)
- [20] Michael E.: On k -spaces, k_R -spaces and $k(X)$. *Pacific J. Math.* **47**, 487–498 (1973)
- [21] Noble N.: Ascoli theorems and the exponential map. *Trans. Amer. Math. Soc.* **143**, 393–411 (1969)
- [22] Pol R.: A remark on a question of T. Banach and S. Gabrielyan, handwritten notes (2015)
- [23] Reznichenko E.: Stratifiability of $C_k(X)$ for a class of separable metrizable X . *Topology Appl.* **155**, 2060–2062 (2008)

T.BANACH: IVAN FRANKO NATIONAL UNIVERSITY, LVIV, UKRAINE, AND JAN KOCHANOWSKI UNIVERSITY, KIELCE, POLAND
E-mail address: t.o.banakh@gmail.com

S. GABRIELYAN: DEPARTMENT OF MATHEMATICS, BEN-GURION UNIVERSITY OF THE NEGEV, BEER SHEVA, ISRAEL
E-mail address: saak@math.bgu.ac.il